Homework #7 of Topology II
Due Date: April 23, 2018

1. Show that $r : \mathbb{R}^2 - \{0\} \to S^1$ given by $r(x) = x/\|x\|$ is a deformation retraction.

2. Compute the cohomology of the n-sphere S^n. Cover S^n by two open subsets U and V where U is slightly larger than the northern hemisphere and V slightly larger than the southern hemisphere. Then $U \cap V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$ where S^{n-1} is the equator. Using the Mayer-Vietoris sequence, show that

$$H^*(S^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, n \\ 0 & \text{otherwise.} \end{cases}$$

3. Volume form on a sphere. Let $S^n(r)$ be the sphere of radius r

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$$

in $\mathbb{R}^{n+1}$, and let

$$\omega = \frac{1}{r} \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

(a) Write S^n for the unit sphere. Compute the integral $\int_{S^n} \omega$ and conclude that ω is not exact.

(b)Regarding r as a function on $\mathbb{R}^{n+1} - 0$, show that $dr\omega = dx_1 \cdots dx_{n+1}$. Thus ω is the Euclidean volume form on the sphere $S^n(r)$.

4. Compute the cohomology groups $H^*(M)$ and $H_c^*(M)$ of the open Möbius strip M, i.e., the Möbius strip without the bounding edge.[Hint: Apply the Mayer-Vietoris sequence.]

5. Let $f \colon M \to N$ be a continuous map between two manifolds. If f is a proper map, prove that the image of f is closed.

6. (Exercise 5.5 on Page 44 of Bott&Tu) Prove the Five Lemma.

7. Künneth Formula for compact cohomology. The Künneth formula for compact cohomology states that for any manifolds M and N having a finite good cover,

$$H_c^*(M \times N) = H_c^*(M) \otimes H_c^*(N).$$

(a) In case M and N are orientable, show that this is a consequence of Poincare duality and the Künneth formula for de Rham cohomology.

(b) Using the Mayer-Vietoris argument prove the Künneth formula for compact cohomology for any M and N having a finite good cover.